Double Turán Problems

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1 Introduction

This thesis focuses on a variation of the *Turán problem* in extremal combinatorics. The fundamental question in extremal hypergraph theory is determining the maximum number of edges in an n-vertex r-uniform graph that does not contain a prescribed r-uniform graph F as a subgraph. These maxima, denoted $\mathrm{ex}(n, F)$, are referred to as the *extremal numbers* or *Turán numbers* for F. One of the cornerstones of extremal graph theory, concerning the case F is a clique, is Turán's Theorem [19]. To state the theorem, we need the *Turán graphs* $T_k(n)$, which denotes a complete multipartite graph with n vertices and k parts of size $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$.

Theorem A (Turán's Theorem)**.** *The maximum number of edges in an n-vertex graph G containing no clique of order $r+1$ is $e(T_r(n))$, with equality only if $G = T_r(n)$.*

Simonovits [5] observed via the Erdős-Stone Theorem [3] that the asymptotic value of $\mathrm{ex}(n, F)$ may be obtained whenever F is non-bipartite:

Theorem B (Erdős-Stone Theorem, Simonovits' Theorem)**.** *Let F be any graph of chromatic number $r + 1 \geq 3$. Then $\mathrm{ex}(n, F) = (1 + o(1))T_r(n)$ as $n \to \infty$.*

There are a number of proofs of the Erdős-Stone Theorem. A very general framework involves *Szemerédi's Regularity Lemma*, which may be stated as follows. A pair (U, V) of disjoint sets of vertices in a graph G is called *ϵ-regular* if for any $X \subseteq U$ and $Y \subseteq V$ of size at least $\epsilon|U|$ and $\epsilon|V|$ respectively,

$$\left| \frac{e(X, Y)}{|X||Y|} - \frac{e(U, V)}{|U||V|} \right| < \epsilon.$$

The following was proved by Szemerédi [18]:

Theorem C (Szemerédi's Regularity Lemma)**.** *For all $\epsilon > 0$, there exist m and M such that for every graph G, there exists a partition $(V_1, V_2, \dots, V_k)$ of $V(G)$ such that $m \leq k \leq M$ and $|V_1| \leq |V_2| \leq \cdots \leq |V_k| \leq |V_1| + 1$ and all but at most ϵk^2 pairs (V_i, V_j) are ϵ-regular.*

The value of $\mathrm{ex}(n, F)$ for bipartite F is in general wide open, and the order of magnitude of $\mathrm{ex}(n, K_{4,4})$ or $\mathrm{ex}(n, C_8)$ is not known – see Furedi and Simonovits [8] for a history of the bipartite Turán problem. There is also no analog of the above theorems for r-uniform hypergraphs. The asymptotic value of $\mathrm{ex}(n, K_k^r)$ is not known for any $k > r \geq 3$, where K_k^r denotes the complete r-uniform hypergraph on k vertices. The asymptotic value of $\mathrm{ex}(n, K_4^3)$ was conjectured by Turán [19] to be $\frac{5}{9}\binom{n}{3}$, and this remains open despite decades of intensive research.

1.1 Problem Statement

In this thesis, we investigate closely related problems which we refer to as *double Turán problems*. To describe these problems, let $G_1, G_2, \ldots, G_m$ be graphs with the same vertex set $V(G_i) = [n]$ for $i \in [m]$.

Definition 1. *For a graph F, we say that $G_1, G_2, \ldots, G_m$ is double F-free if $E(F) \nsubseteq E(G_i) \cap E(G_j)$ for $1 \le i < j \le m$. Moreover, we call a copy of F in $G_i \cap G_j$ a double F in G_i, G_j.*

The double Turán problem asks for the maximum sum of edges over $G_1, G_2, \ldots, G_m$ such that $G_1, G_2, \ldots, G_m$ are double F-free.

Definition 2. *For a graph F, the double Turán number for F, denoted $\phi(m, n, F)$, is the maximum value of $\sum_{i=1}^{m} e(G_i)$ such that $G_1, G_2, \ldots, G_m$ are double F-free.*

We would also like to study a special case of the double Turán problem, which we refer to as *induced double Turán problem.*

Definition 3. *We call $G_1, G_2, \ldots, G_m$ induced if for each $i \in [m]$, G_i is an induced subgraph of $\bigcup_{i=1}^{m} G_i$.*

In other words, if $\{u, v\} \in E(G_i)$ and $u, v \in V(G_j)$, then $\{u, v\} \in E(G_j)$.

Definition 4. *For a graph F, the induced double Turán number, denoted $\phi^*(m, n, F)$, is the maximum value of $\sum_{i=1}^{m} e(G_i)$ such that $G_1, G_2, \ldots, G_m$ are induced and double F-free.*

The induced double Turán problem is then to determine $\phi^*(m, n, F)$. Clearly, $\phi(m, n, F) \ge \phi^*(m, n, F)$. Similar to the Turán problem, the induced and non-induced double Turán problems behave differently depending on whether F is bipartite or non-bipartite. Thus we will study these two cases separately in this thesis. We shall see that the study of $\phi(m, n, F)$ and $\phi^*(m, n, F)$ is motivated by certain hypergraph extremal problems.

1.2 Motivation : Link Graphs and Hypergraphs

Apart from the intrinsic interest in investigating $\phi(m, n, F)$, one motivation to study is that $\phi(m, n, F)$ is closely connected to pure hypergraph extremal problems via the notion of *link graphs*. Let H be a triple system, that is, a set of three-element subsets of a finite set $[n]$. We may view H as a 3-uniform hypergraph, where the edges are the three-element subsets of $[n]$.

Definition 5. *For $i \in V(H)$, the link graph of i, denoted H_i, is the graph with $V(H_i) = V(H)\backslash\{i\}$ and $E(H_i) = \{\{j,k\} : \{i,j,k\} \in E(H)\}$.*

A handy idea in extremal hypergraph theory is to reduce a hypergraph extremal problem to extremal problems for the link graphs. For instance, a triple system H does not contain a tetrahedron, i.e. four triples on four vertices, if and only if all its link graphs are triangle-free.

In the current context, given a graph F, let F^+ denote the triple system with $V(F^+) = V(F) \cup \{x,y\}$ and $E(F^+) = \{e \cup \{x\}, e \cup \{y\} : e \in E(F)\}$. For example, if F is a 4-cycle $K_{2,2}$, then F^+ is the hypergraph for octahedron. We can see that $\phi(n,n,F)$ and $\mathrm{ex}(n,F^+)$ are intimately related: if H is an F^+-free triple system with vertex set $[n]$, then the link graphs $H_1, H_2, \ldots, H_n$ are double F-free, otherwise there exists a double F in some $H_i \cap H_j$ and the vertices i,j along with that copy of F form a F^+ in H, contradiction. Thus, $\mathrm{ex}(n,F^+) \le \phi(n,n,F)$, and this relates the double Turán problem to hypergraph extremal problems.

On the other hand, the study of the induced double Turán problem is motivated by a special case of the *generalized Turán problem*, which asks for the maximum number $\mathrm{ex}(n,\bar{F},K_3)$ of triangles in a graph G with vertex set $[n]$ that does not contain some graph $\bar{F}$. This problem was studied by Alon and Shikhelman [1] and Kostochka, Mubayi and Verstraete [10, 12, 14]. Similar to how link graphs relate to hypergraph extremal problems, the generalized Turán problem is related to $\phi^*(n,n,F)$ as follows: Let $\bar{F}$ be the graph consisting of all pairs contained in triples in F^+. For example, if $F = K_{2,2}$ then $\bar{F} = K_{2,2,2}$. For $i \in [n]$, define $E(G_i) = \{\{j,k\} : \{i,j\},\{j,k\},\{i,k\} \in E(G)\}$. Then $G_1, G_2, \ldots, G_n$ are induced and double F-free, so $\phi^*(n,n,F) \ge \mathrm{ex}(n,\bar{F},K_3)$. This relates the induced double Turán problem to extremal problems for triangles in graphs.

1.3 Main Results : The Induced Case

The determination of $\phi^*(m,n,F)$ turns out to be fairly straightforward when F is a non-bipartite graph: the extremal objects are simply m copies of the same extremal graph for F.

Theorem 1. *Let $r, m \ge 3$. There exists $n_0(r)$ such that if $n \ge n_0(r)$ and F is a graph of chromatic number r, then*

$$\phi^*(m,n,F) = m \cdot \mathrm{ex}(n,F).$$

Moreover, $G_1, \ldots, G_m$ are induced double F-free graphs on $[n]$ that sum up to $m \cdot \mathrm{ex}(n,F)$ edges only if $G_1 = \cdots = G_m$ are identical extremal n-vertex F-free graphs.

In the case $F = K_r$, we shall see the theorem is true for all $n \geq 3$:

Theorem 2. *Let $m, n, r \geq 3$. Then*

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)).$$

Moreover, $G_1, \dots, G_m$ are induced double K_r-free graphs on $[n]$ that sum up to $m \cdot e(T_{r-1}(n))$ edges only if $G_1 = \dots = G_m = T_{r-1}(n)$.

In the case F is a bipartite graph, even determining the order of magnitude of $\phi^*(m, n, F)$ appears to be difficult. In fact, we do not even know the order of magnitude of $\phi^*(m, n, P)$ when P is a path with two edges. In this thesis, we propose the following very broad conjecture:

Conjecture A. *Let F be any non-empty graph and $m, n \geq 1$. Then*

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

It is clear that a single complete graph K_n does not contain a double F, and neither do identical copies $G_1, G_2, \dots, G_m$ of an extremal n-vertex F-free graph. Thus we have the trivial lower bound

$$\phi^*(m, n, F) \geq \max\left\{\binom{n}{2}, m \cdot \mathrm{ex}(n, F)\right\}.$$

This conjecture is true when F is non-bipartite, by Theorem 1. If F is bipartite, then the upper bounds on $\phi^*(m, n, F)$ are more difficult to come by, especially when m is large. For instance, from our discussion in the previous section, we know

$$\mathrm{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2}),$$

and so Conjecture A implies that an n-vertex graph not containing the octahedron graph has $O(n^{5/2})$ triangles. In fact, it is also the case that $\mathrm{ex}(2n, K_{2,2,2}, K_3) \geq \phi^*(n, n, K_{2,2})$: if we have double $K_{2,2}$-free induced graphs $G_1, G_2, \dots, G_n$ with vertex set $[n]$, then let H be the graph with $V(H) = [2n]$ consisting of all triangles with vertex set $\{i, j, k\}$ such that $n < k \leq 2n$ and $\{i, j\} \in E(G_k)$. The graph H is $K_{2,2,2}$-free and $|E(H)| = \sum_{i=1}^{n/2} e(G_i)$. This shows that $\mathrm{ex}(n, K_{2,2,2}, K_3)$ and $\phi^*(n, n, K_{2,2})$ are equivalent up to a constant factor. Similarly, we have

$$\mathrm{ex}(n, K_{1,2,2}, K_3) \leq \phi^*(n, n, K_{1,2}),$$

and so Conjecture A implies that an n-vertex graph not containing the wheel graph has $O(n^2)$ triangles, which is conjectured by Mubayi and Verstraete [14]. The conjecture proposes more generally that if F is a tree, then $\phi^*(n,n,F) = O(n^2)$. In fact, it is possible to prove the following theorem using the *removal lemma* as in [12] as well as a projective plane construction for $\phi(n,n,P)$:

Theorem 3. *Let P be a path with two edges. Then $\phi(n,n,P) = \Omega(n^{5/2})$, whereas $\phi^*(n,n,P) = o(n^{5/2})$, as $n \to \infty$. In particular,*

$$\lim_{n\to\infty} \frac{\phi^*(n,n,P)}{\phi(n,n,P)} = 0.$$

Apart from deterimining the order of magnitude of $\phi^*(m,n,F)$, this above theorem also shows that the order of magnitude of $\phi(n,n,P)$ and $\phi^*(n,n,P)$ can differ significantly. This suggests that the induced and non-induced double Turán problems may be fundamentally different.

A special bipartite case where Conjecture A is true is when F is a matching with two edges. Let M denote a matching with two edges, and let M^+ denote the graph obtained from two copies of K_4 sharing one edge by removing that edge. Then $\mathrm{ex}(n, M^+, K_3) \le \phi^*(n,n,M)$. If F is the triple system consisting of all four triangles in M^+, then Furedi [7] showed $\mathrm{ex}(n, M^+) = O(n^2)$, answering a conjecture of Erdős [4]. It is possible to adapt Furedi's proof to give $\phi^*(n,n,M) = O(n^2)$, so in this case Conjecture A is true and $\mathrm{ex}(n, M^+, K_3) = \Theta(\phi^*(n,n,M))$. For improvements of the constant factor, see Mubayi and Verstraete [13] and Pikhurko and Verstraete [15]. We shall see that for some bipartite F, if m is not too large relative to n, then Conjecture A is also true.

1.4 Main Results : The Non-induced Case

Determining $\phi(m,n,F)$ even when F is a complete graph is challenging. The forth theorem we give is well-suited to the case of certain bipartite graphs, and is due to Wilson:

Theorem 4. *Let F be a graph. If there exists an extremal F-free n-vertex graph with maximum degree at most $n^{1/2}/m^2$, then*

$$\phi(m,n,F) = \binom{n}{2} + \binom{m}{2}\mathrm{ex}(n,F).$$

By the Erdős-Stone Theorem, the extremal number for any non-bipartite F is $\Theta(n^2)$, which implies that the maximum degree of an extremal F-free graph exceeds $n^{1/2}$. Thus the condition on the maximum degree in the above theorem can only be satisfied for bipartite graphs, for instance, a path of two edges. An example of a bipartite graph that does not apply to this theorem would be a matching of two edges, whose extremal graph has a maximum degree of $n-1$.

Since $\binom{n}{2}+m-1 \le \phi^*(m,n,F) \le \phi(m,n,F)$ for any graph F with at least two edges, Theorem 4 shows $\phi^*(m,n,F)=(1+o(1))\binom{n}{2}$ whenever the conditions on m in the theorem are satisfied, proving Conjecture A to be true in this case.

Our first theorem on $\phi(m,n,F)$ for non-bipartite graphs F uses the notion of *supersaturation* – see Erdős and Simonovits [6]. We determine the asymptotic value of $\phi(m,n,F)$ as $m\to\infty$ when F is a non-bipartite graph:

Theorem 5. *Let $n \ge 1$ and let F be a non-bipartite graph. Then as $m \to \infty$,*

$$\phi(m,n,F) = (1+o(1))m \cdot \mathrm{ex}(n,F).$$

The next result we present concerns non-bipartite graphs. To state the theorem, we require the notion of the *M-color Ramsey number.*

Definition 6. *For $M \ge 2$, the M-color Ramsey number, denoted $R_M(r)$, is the smallest integer N such that any coloring of the edges of the complete graph K_N with M colors contains a monochromatic complete subgraph on r vertices.*

Suppose we have a monochromatic K_r-free coloring $c : E(K_N) \to 2^{[m]}$. For $i \in [m]$, let $H_i = \{\{u,w\} \in E(K_N) : i \in c(u,w)\}$. Then $H_1, H_2, \dots, H_m$ are double K_r-free. If we replace the vertices of K_N with disjoint sets $V_1, V_2, \dots, V_N$ whose sizes add up to n, and then let

$$G_i = \{\{x,y\} : (x,y) \in V_u \times V_w,\, i \in c(u,w),\, 1 \le u < w \le N\}$$

and make each V_i cliques in G_1, then $G_1, G_2, \dots, G_m$ is also double K_r-free. We call $G_1, G_2, \dots, G_m$ an (m,n,N)-blowup.

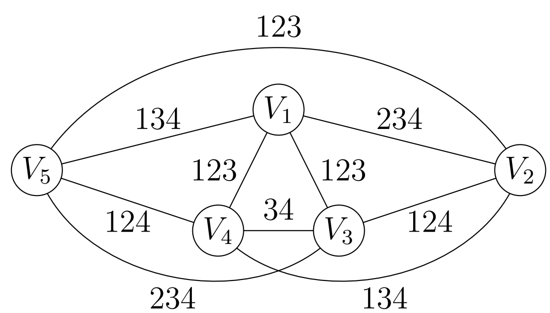

Figure 1: Example of an $(4,n,5)$-blowup not containing a double K_3.

Definition 7. *Let $f(m,n,r)$ denote the maximum of $e(G_1)+e(G_2)+\cdots+e(G_m)$ such that $G_1, G_2, \ldots, G_m$ is a double K_r-free (m,n,N)-blowup for some $N < R_{\binom{m}{2}}(r)$.*

This turns out to be exactly the construction which determines $\phi(m,n,F)$ when F is a complete graph:

Theorem 6. *For $m, n, r \geq 3$,*

$$\phi(m,n,K_r) = f(m,n,r).$$

While computing $f(m,n,r)$ is a finite calculation, the Ramsey number $R_{\binom{m}{2}}(r)$ unfortunately appears to be intractable in general. It is known that $R_2(3) = 6$ and $R_3(3) = 17$ and $R_2(4) = 18$, but no further multicolor Ramsey numbers are known [2, 11]. In the special case $r = m = 3$, the following holds:

Theorem 7. *For $n \geq 1$,*

$$\phi(3,n,K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

The same problem immediately becomes difficult when m is increased by 1. The blowup construction in Figure 1 shows $\phi(4,n,K_3) - [\binom{n}{2} + 3\mathrm{ex}(n,K_3)] \geq n^2/100$ as $n \to \infty$. This suggests the actual values of $\phi(m,n,K_r)$ can be significantly larger than our trivial lower bound. We leave it as an open problem to determine $\phi(m,n,K_r)$ for $r, m \geq 3$ and $(r,m) \neq (3,3)$.

1.5 Definitions and Notations

Denote the set of first n positive integers as $[n] = \{1,2,\ldots,n\}$. Given a set X, we denote 2^X as the power set of X. Given graph $G = (V,E)$, let $V(G)$ denote the vertex set and $E(G)$ denote the edge set of G. Let $e(G) = |E(G)|$ be the number of edges in G. For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u,v\} \in E(G)\}$ the neighborhood of v. Given two graphs G_1, G_2, we denote $G_1 \cup G_2$ as the graph on $V(G_1) \cup V(G_2)$ with edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Similarly, we define $G_1 \cap G_2$ as the graph on $V(G_1) \cap V(G_2)$ with edge set $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$. In this thesis, we reserve n to denote the number of vertices in a graph. We call a n-vertex complete graph K_n, and a complete bipartite graph $K_{a,b}$, where a, b are the sizes of its parts. Given graph G, H, define $G + H$ as the graph fully connecting G, H, i.e. $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{\{u,v\} : u \in V(G), v \in V(H)\}$. Let v be a vertex from $G_1, G_2, \ldots, G_m$. Unless otherwise specified, we denote $d(v)$ as the sum of the degree of v over all G_i.

2 The Induced Double Turán Problem

We prove the theorems for $\phi^*(m,n,F)$ in this chapter. In particular, the main theorem we prove is Theorem 1 for general non-bipartite graphs F and in the special case of cliques. We will first introduce two observations that simplify the problem.

The first observation is that the determination of $\phi^*(m,n,F)$ can be reduced to smaller values of m:

Lemma 8. *Let $n,m,k \geq 2$ with $m \geq k$, and let F be some graph. Then*

$$\phi^*(m,n,F) \leq \frac{m}{k} \cdot \phi^*(k,n,F).$$

Moreover, let $G_1,\dots,G_m$ be induced double F-free graphs on $[n]$ and suppose $\sum_{i=1}^k e(G_i) = \phi^*(k,n,F)$ only if $G_1 = \cdots = G_k$. Then $\sum_{i=1}^m e(G_i) = \phi^*(m,n,F)$ only if $G_1 = \cdots = G_m$.

Proof. Let $G_1,\dots,G_m$ be induced double F-free graphs on $[n]$. Put $G_{i+m} = G_i$ for all $i \in [m]$. Then

$$\sum_{i=1}^m e(G_i) = \frac{1}{k}\sum_{i=1}^m [e(G_i) + \cdots + e(G_{i+k-1})] \leq \frac{1}{k}\sum_{i=1}^m \phi^*(k,n,F) = \frac{m}{k} \cdot \phi^*(k,n,F),$$

which establishes the upper bound. The lower bound follows from the construction with $G_1 = \cdots = G_m$ to be n-vertex extremal graphs for F.

Now suppose $\sum_{i=1}^m e(G_i) = (m/k)\phi^*(k,n,F)$ and $G_1 \neq G_2$. By assumption $\sum_{i=1}^k e(G_i) < \phi^*(k,n,F)$. But then $\sum_{i=1}^k e(G_{i+j}) > \phi^*(k,n,F)$ for some $j \geq 1$, contradiction. □

The second observation shows that the determination of $\phi^*(2,n,F)$ can be reduced to an optimization problem over the number of vertices in the intersection of the two graphs:

Lemma 9. *Let $n \geq 1$. For graph F, define $\mathcal{C}(n,t,F) := \binom{n-t}{2} + (n-t)t + 2\mathrm{ex}(t,F)$. Let G_1, G_2 be induced double F-free graphs on $[n]$. Then*

$$e(G_1) + e(G_2) \leq \max_{0 \leq t \leq n} \mathcal{C}(n,t,F),$$

with equality only if G_2 is an extremal graph for F with t_{max} vertices and $G_1 = G_2 + K_{n-t}$, where t_{max} is a maximizer of $\mathcal{C}(n,t,F)$ over $0 \leq t \leq n$.

Proof. Let G_1, G_2 be induced double F-free graphs on $[n]$. Put $T = V(G_1) \cap V(G_2)$, $t = |T|$, $s = |V(G_1) \backslash T|$, and $n-t-s = |V(G_2) \backslash T|$. Note that $t, s \in \mathbb{Z}_{\geq 0}$. Since G_1, G_2 are induced subgraphs of $G_1 \cup G_2$, we have $G_1[T] = G_2[T] = G_1 \cap G_2$. But then $G_1 \cap G_2$ is F-free, so $e(G_1[T]) = e(G_2[T]) \leq \mathrm{ex}(t, F)$. Notice there can be at most $t(n-t)$ edges between T and $(V(G_1) \cup V(G_2)) \backslash T$. Since $G[V(G_1) \backslash T] \leq \binom{s}{2}$ and $G[V(G_2) \backslash T] \leq \binom{n-t-s}{2}$,

$$e(G_1) + e(G_2) \leq \binom{s}{2} + \binom{n-s-t}{2} + t(n-t) + 2\mathrm{ex}(t, F).$$

But then $\binom{n-t}{2} > \binom{s}{2} + \binom{n-t-s}{2}$ for $0 < s < n-t$, so

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2\mathrm{ex}(t, F) = \mathcal{C}(n, t, F).$$

This establishes the upper bound. From this we also know that $e(G_1) + e(G_2) = \mathcal{C}(n, t, F)$ only if G_2 is the t-vertex extremal graph for F and $G_1 = G_2 + K_{n-t}$. The result now follows. $\square$

2.1 Proof of Theorem 2

By Lemma 8, it suffices to prove the theorem for $m = 3$. Let G_1, G_2, G_3 be induced double K_r-free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, K_r)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\phi^*(3, n, K_r) \geq 3\mathrm{ex}(n, K_r)$. Consequently, we must have $e(G_1) + e(G_2) \geq 2\mathrm{ex}(n, K_r)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\mathrm{ex}(n, K_r)$, it suffices to show that $G_1 = G_2 = T_{r-1}(n)$. In particular, we will use Lemma 9 to show that G_1, G_2 is an extremal configuration without containing a double K_r.

Let $t = |V(G_1 \cap G_2)|$. By Turán's Theorem,

$$\mathrm{ex}(t, K_r) - \mathrm{ex}(t-1, K_r) = e(T_{r-1}(t)) - e(T_{r-1}(t-1)) = t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, K_r) - \mathcal{C}(n, t-1, K_r) = -t + 1 + 2[\mathrm{ex}(t, K_r) - \mathrm{ex}(t-1, K_r)] = t + 1 - 2\left\lceil \frac{t}{r-1} \right\rceil. \quad (1)$$

For $r \geq 4$, $\mathcal{C}(n, t, K_r)$ is strictly increasing on t, so by Lemma 9,

$$\phi^*(2, n, K_r) = \mathcal{C}(n, n, K_r) = 2\mathrm{ex}(n, K_r) = e(G_1) + e(G_2)$$

and $G_1 = G_2 = T_{r-1}(n)$, as desired.

Now suppose $r = 3$. Equation (1) shows that $\mathcal{C}(n, t, K_r)$ is non-decreasing on t and

$\mathcal{C}(n,t,K_r) > \mathcal{C}(n,t,K_r)$ for even t. By Lemma 9, we now have

$$\phi^*(2,n,K_r) = \max[\mathcal{C}(n,n,K_r), \mathcal{C}(n,n-1,K_r)] = 2\mathrm{ex}(n,K_r) = e(G_1) + e(G_2),$$

and either $G_1 = G_2 = T_{r-1}(n)$, or $G_2 = T_{r-1}(n-1)$ and $G_1 = G_2 + K_1$. If the latter case is true, then $e(G_3) \geq \mathrm{ex}(n,F) > e(G_2)$, and this contradiction completes the proof. □

2.2 Proof of Theorem 1

If F is a graph of chromatic number $r+1 \geq 3$, then Theorem B shows $\mathrm{ex}(n,F) = (1+o(1))\mathrm{ex}(n,K_{r+1})$ as $n \to \infty$. In this section, we prove Theorem 1 following the same line of reasoning as in the proof of Theorem 2.

Proof of Theorem 1. By Lemma 8, it suffices to prove the theorem for $m = 3$. Let G_1, G_2, G_3 be induced double F-free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \phi^*(3,n,F)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\phi^*(3,n,F) \geq 3\mathrm{ex}(n,F)$. Consequently, we must have $e(G_1) + e(G_2) \geq 2\mathrm{ex}(n,F)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\mathrm{ex}(n,F)$, it suffices to show that $G_1 = G_2$ are n-vertex F-free extremal graphs. In particular, we will use Lemma 9 to show that G_1, G_2 is an extremal configuration without containing a double F.

Let $t = |V(G_1 \cap G_2)|$. If $t < \sqrt{n}$, then

$$2\mathrm{ex}(n,F) \geq 2e(T_{r-1}(n)) \geq 2\left\lfloor \frac{n^2}{4} \right\rfloor \geq \binom{n}{2} + \binom{\sqrt{n}}{2} > \mathcal{C}(n,t,F).$$

Thus $t \geq \sqrt{n}$. But then for large enough t, any extremal t-vertex F-free graph contains a spanning complete $(r-1)$-partite subgraph $T_{r-1}(t)$, so we may add $\mathrm{ex}(t-1,F) - e(T_{r-1}(t-1))$ egdes to $T_{r-1}(t)$ and still avoid F as a subgraph. Hence for large enough t, we have $\mathrm{ex}(t,F) \geq \mathrm{ex}(t-1,F) - e(T_{r-1}(t-1)) + e(T_{r-1}(t))$, and so

$$\mathrm{ex}(t,F) - \mathrm{ex}(t-1,F) \geq e(T_{r-1}(t)) - e(T_{r-1}(t-1)) \geq t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n,t,F) - \mathcal{C}(n,t-1,F) = -t + 1 + 2[\mathrm{ex}(t,F) - \mathrm{ex}(t-1,F)] \geq t + 1 - 2\left\lceil \frac{t}{r-1} \right\rceil. \quad (2)$$

For $r \geq 4$, $\mathcal{C}(n, t, F)$ is strictly increasing on t, so by Lemma 9,

$$\phi^*(2, n, F) = \mathcal{C}(n, n, F) = 2\mathrm{ex}(n, F) = e(G_1) + e(G_2),$$

and $G_1 = G_2$ are n-vertex F-free extremal graphs, as desired.

Now suppose $r = 3$. Equation (2) shows that $\mathcal{C}(n, t, F)$ is strictly increasing for even t and $\mathcal{C}(n, t, F) \geq \mathcal{C}(n, t-1, F)$ for odd t. By Lemma 9, we now have

$$\phi^*(2, n, F) = \max[\mathcal{C}(n, n, F), \mathcal{C}(n, n-1, F)] = 2\mathrm{ex}(n, F) = e(G_1) + e(G_2),$$

and either $G_1 = G_2$ are n-vertex extremal F-free graphs, or G_2 is an $(n-1)$-vertex extremal F-free graph and $G_1 = G_2 + K_1$. If the latter case is true, then $e(G_3) \geq \mathrm{ex}(n, F) > e(G_2)$, and this contradiction completes the proof. $\square$

2.3 Proof of Theorem 3

We need to show that $\phi(n, n, P) = \Omega(n^{5/2})$ and $\phi^*(n, n, P) = o(n^{5/2})$, as $n \to \infty$.

Claim 1. *For $\sqrt{n} < m \leq n$,*

$$\phi(n, n, P) = (1/2 + o(1))mn^{3/2},$$

as $n \to \infty$.

We first show that $\phi(m, n, P) \leq (mn^{3/2} + n^2)/2$. For each vertex $u \in [n]$, define H_u as the $m \times n$ bipartite graph with edge set $E(H_u) \coloneqq \{\{v, i\} : \{u, v\} \in E(G_i)\}$. If H_u contains a quadrilateral $\{v, i\}, \{v, j\}, \{w, i\}, \{w, j\}$, then $\{u, v\}, \{u, w\}$ form a double P in $G_i \cap G_j$, contradiction. Thus we conclude that H_u is quadrilateral-free, and therefore $e(H_u) \leq m\sqrt{n} + n$, by the Kővári-Sós-Turán Theorem [9]. It now follows that

$$\sum_{i=1}^{m} e(G_i) = \frac{1}{2} \sum_{u \in V(G)} e(H_u) \leq \frac{1}{2}(mn^{3/2} + n^2).$$

We now show the upperbound is tight asymptotically by giving a finite projective plane construction. Suppose $G_1, G_2, \ldots, G_n$ are graphs on $[n]$ containing no double P and $\sum_{i=1}^{n} e(G_i) \geq (1/2 + o(1))n^{5/2}$, with $e(G_1) \geq e(G_2) \geq \cdots \geq e(G_n)$. Then $G_1, G_2, \ldots, G_m$ are graphs with no double P and $\sum_{i=1}^{m} e(G_i) \geq (1/2 + o(1))mn^{3/2}$. Hence, it suffices to prove the case for $m = n$.

Consider a finite projective plane with n points and n lines, with prime q chosen so that $n = (1 + o(1))(q^2 + q + 1)$ as $q \to \infty$. Let $S_1, \ldots, S_n \subseteq [n]$ be the n lines of the projective plane. Note that each line S_i contains $q + 1$ points, and the intersection of any two distinct lines S_i, S_j contains $|S_i \cap S_j| = 1$ point.

Define $G_1, \ldots, G_n$ to be graphs on $[n]$, each with edge set

$$E(G_i) \coloneqq \{\{j,k\} \subseteq [n] : j \neq k,\, j + k \in S_i \mod n\}.$$

Note that the intersection of distinct G_i, G_j is P free: since $|S_i \cap S_j| = 1$, if $\{a,b\}, \{a,c\} \in E(G_i) \cap E(G_j)$, then $a + b = a + c$ so $b = c$.

We now count the number of edges in $G_1, \ldots, G_n$. Since $|S_i| = q + 1$, for each point $j \in [n]$, there are $q + 1$ choices for $k \in [n]$ such that $j + k \in S_i$. But then we have to avoid counting the same edge twice and loops, so the number of edges in G_i is

$$e(G_i) = \frac{n(q+1) - \#\text{loops counted for } G_i}{2}.$$

If $j \in [n]$ is even, then $k = j/2$ is the unique number in $[n]$ such that $k + k = j \mod n$. If $j \in [n]$ is odd, then $k = (n+j)/2$ is the unique number in $[n]$ such that $k + k = j \mod n$, as n is even. Hence, for each $j \in S_i$, there exists a unique $k \in [n]$ such that $k + k = j \mod n$, and thus

$$\#\text{loops counted for } G_i = |S_i| = q + 1.$$

Since $q + 1 = (1 + o(1))n^{1/2}$, the number of edges in $G_1, \ldots, G_n$ is

$$\sum_{i=1}^{n} e(G_i) = n \cdot \frac{n(q+1) - (q+1)}{2} = \left(\frac{1}{2} + o(1)\right) n^{5/2},$$

as $n \to \infty$. This proves the first claim.

Claim 2. *$\phi^*(n, n, P) = o(n^{5/2})$, as $n \to \infty$.*

Let $G_1, G_2, \ldots, G_n$ be induced and double P-free and let $\epsilon > 0$. Let $d_i(v)$ be the degree of vertex v in the graph G_i. Let I be the set of pairs (i,v) such that $d_i(v) \geq \sqrt{n}/\epsilon + 1$. Since $G_1, G_2, \ldots, G_n$ do not contain a double P,

$$\sum_{(i,v) \in I} \binom{d_i(v)}{2} \leq n^3.$$

The maximum possible value of $\sum_{(i,v)\in I} d_i(v)$ subject to this constraint is when $d_i(v) =$

$\sqrt{n}/\epsilon + 1$ for all (i,v), in which case $|I| \leq 2\epsilon^2 n^2$ and so

$$\sum_{(i,v)\in I} d_i(v) \leq (2\epsilon^2 n^2) \cdot \left(\frac{\sqrt{n}}{\epsilon} + 1\right) = 3\epsilon n^{5/2}$$

for large enough n. Remove all edges of G_i on vertex v such that $(i,v) \in I$. The total number of edges removed is at most $3\epsilon n^{5/2}$. Let $G'_1, G'_2, \ldots, G'_n$ be the remaining subgraphs of $G_1, G_2, \ldots, G_n$. If $e(G'_i) \leq \epsilon n^{3/2}$, then remove all edges of G'_i. The number of edges removed in this process is at most $\epsilon n^{5/2}$. The remaining graphs $G''_1, G''_2, \ldots, G''_m$ have each at least $\epsilon n^{3/2}$ edges and maximum degree at most $\sqrt{n}/\epsilon$. In particular, each G''_i contains a matching M_i of size at least $e(G''_i)/2\Delta(G''_i) = \epsilon^2 n/2$. If $m \leq \epsilon n$, then

$$\sum_{i=1}^{n} e(G_i) \leq 4\epsilon n^{5/2} + \sum_{i=1}^{m} e(G''_i) \leq 4\epsilon n^{5/2} + \phi(m,n,P) \leq 5\epsilon n^{5/2}$$

by Claim 1. If $m > \epsilon n$, then we apply Szemerédi's Regularity Lemma to find, for some $\delta > 0$ depending only on ϵ, a matching M_1 in G''_1 such that for some pair of set $X, Y \subseteq V(M_1)$ of size at least δn each, there is a set E of at least $\delta^3 n^2$ edges $\{x,y\}$ of $G''_1 \cup G''_2 \cup \cdots \cup G''_m$ such that $x \in X$ and $y \in Y$. Since G''_1 is induced, $E \subseteq E(G_1)$. In particular, there are at least $\delta^5 n^3/4$ copies of P in G_1. We can repeat the argument in the remaining graphs $G''_i : i \in [2,m]$ to get say M_2 in G''_2 as above, which gives $\delta^5 n^3/4$ copies of P in G_2. If we do this $4\delta^{-5}$ times, then we have found n^3 copies of P in the first $4\delta^{-5}$ graphs, and two of them have the same edge-set. We conclude $\sum_{i=1}^{n} e(G_i) \leq 5\epsilon n^{5/2}$ if n is large enough. Since ϵ is arbitrary, this completes the proof. $\square$

3 The Non-induced Double Turán Problem

In this section, we prove our main theorems on $\phi(m,n,F)$.

3.1 Proof of Theorem 4

We first show that for all $m,n \geq 1$ and graph F,

$$\phi(m,n,F) \leq \binom{n}{2} + \mathrm{ex}(n,F)\binom{m}{2}.$$

Thereafter, we show that if there is an extremal F-free graph with maximum degree at most $n^{1/2}/m^2$, then the above bound is tight.

Proof of the upper bound. For $S \subseteq [m]$, let E_S denote the set of edges that are contained in exactly $\{G_i\}_{i\in S}$, and note that $E_S \cap E_{S'} = \emptyset$ if $S \neq S'$. Then

$$\sum_{i=1}^{m} e(G_i) = \sum_{S\subseteq[m]} |S||E_S| \leq \binom{n}{2} + \sum_{S\subseteq[m],|S|\geq 2} (|S|-1)|E_S|.$$

Note that $\bigsqcup_{T\supseteq S} E_T = \bigcap_{i\in S} E(G_i)$, which is F-free for $|S| \geq 2$ and so

$$\sum_{T\supseteq S} |E_T| \leq \mathrm{ex}(n,F).$$

It now follows that

$$\sum_{\substack{S\subseteq[m]\\|S|\geq 2}} (|S|-1)|E_S| = \sum_{\substack{S\subseteq[m],\\|S|=2}} \sum_{T\supseteq S} \frac{(|T|-1)|E_T|}{\binom{|T|}{2}} \leq \sum_{\substack{S\subseteq[m],\\|S|=2}} \sum_{T\supseteq S} |E_T| \leq \binom{m}{2}\mathrm{ex}(n,F),$$

as each $T \in [m]$ with $|T| \geq 2$ is counted $\binom{|T|}{2}$ times in total and $|T|-1 \leq \binom{|T|}{2}$. This proves the upper bound.

Proof of the lower bound. We need to show there exists a construction such that the graph with edge set E_S is an extremal F-free graph, for all $S \subseteq [m]$ of size 2. Let $M = \binom{m}{2}$ and $H_1, \ldots, H_M$ be copies of an extremal F-free graph on n vertices such that H_i with maximum degree $\triangle \leq n^{1/2}/m^2$ for all $i \in [m]$. It suffices to show that we can embed each H_i onto $[n]$ such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each H_i and iteratively decrease the number of intersecting edges. Define a (u,v,i)-*swap* by swapping the embedding of vertex u and v of H_i, i.e. replacing each edge $\{u,w\} \in E(H_i)$

with the edge $\{u, w\}$ and each edge $\{v, w\} \in E(H_i)$ with the edge $\{v, w\}$. This preserves the type of isomorphism of H_i. Given a vertex v, let $N(v) = N_{H_1}(v) \cup \cdots \cup N_{H_M}(v)$. Suppose there exists an intersecting edge $\{u, w\} \in E(H_i) \cap E(H_j)$. Since $|N(u)| \leq M \cdot \triangle \leq n^{1/2}/2$, $|N(u) \cup N(N(u))| \leq \triangle + \triangle(\triangle - 1) \leq n/4$, so there exists a vertex $v \notin N(u) \cup N(N(u))$. Since $N(u) \cap N(v) = \emptyset$, performing a (u, v, i)-swap reduces the number of intersecting edges. The result now follows from iterating this process. $\square$

3.2 Proof of Theorem 5

We need the following *saturation theorem*, which may be found in [6].

Proposition 10. *Let F be any non-empty graph with k vertices. For all $\epsilon > 0$, there exists $\delta > 0$ such that if G is any n-vertex graph with $\mathrm{ex}(n, F) + \epsilon n^2$ edges, then G contains δn^k copies of F.*

Proof of Theorem 5. Let $k = |V(F)|$ and let $\epsilon > 0$. Let $G_1, G_2, \ldots, G_m$ be double F-free. Reorder $G_1, G_2 \ldots, G_m$ so that $e(G_i) \geq \mathrm{ex}(n, F) + \epsilon n^2$ for $1 \leq i \leq \ell$ and $e(G_i) < \mathrm{ex}(n, F) + \epsilon n^2$ for $\ell < i \leq m$. Then each $G_i : 1 \leq i \leq \ell$ contains at least δn^k copies of F, by Proposition 10. On the other hand, there are at most n^k copies of F such that $F \subseteq G_i$ for some $i \in [m]$. Therefore $\ell \leq 1/\delta$ and

$$\begin{aligned} \sum_{i=1}^{m} e(G_i) &= \sum_{i=1}^{\ell} e(G_i) + \sum_{i=\ell+1}^{m} e(G_i) \\ &\leq \frac{1}{\delta}\binom{n}{2} + (m - \ell)\mathrm{ex}(n, F) + (m - \ell)\epsilon n^2 \\ &\leq m \cdot \mathrm{ex}(n, F) + \epsilon m n^2 + \frac{1}{\delta}\binom{n}{2}. \end{aligned}$$

Since F is not bipartite, $\mathrm{ex}(n, F) = \Theta(n^2)$ and so $\phi(m, n, F) \leq m \cdot \mathrm{ex}(n, F) + (\epsilon + 1/\delta m) m n^2$. Since ϵ was arbitrary and δ is a constant depending only on ϵ, we conclude $\phi(m, n, F) \leq (1 + o(1)) m \cdot \mathrm{ex}(n, F)$ as $m \to \infty$. $\square$

Let F be a bipartite graph with $k \geq 2$ vertices and $j \geq 1$ edges. A strong version of a conjecture of Simonovits [16, 17] would suggest that for all $\epsilon > 0$, there exists $\delta > 0$ such that every n-vertex graph G with at least $p\binom{n}{2}(1 + \epsilon)\mathrm{ex}(n, F)$ edges contains at least $\delta p^j n^k$ copies of F. For instance, this is known to be true whenever the asymptotic behavior of $\mathrm{ex}(n, F)$ is known, which includes the case $F = K_{2,t}$. If F is bipartite and $m \cdot \mathrm{ex}(n, F)/n^2 \to \infty$ as $m, n \to \infty$, then this conjecture with the same proof as above shows $\phi(m, n, F) = (1 + o(1)) m \cdot \mathrm{ex}(n, F)$. When F contains a cycle, then there exists $\alpha > 0$ such that $\mathrm{ex}(n, F) \geq n^{1+\alpha}$ for large enough n. Thus, we conclude that if F contains a cycle and the Simonovits conjecture is true for F, then $\phi(m,n,F) = (1+o(1))m \cdot \mathrm{ex}(n,F)$ for $m \geq n$ and $n \to \infty$. In particular, this shows $\phi(m,n,K_{2,t}) = (1+o(1))m \cdot \mathrm{ex}(n,F)$ for $m \geq n$ as $n \to \infty$.

3.3 Proof of Theorem 6

We now prove Theorem 6. Notice that we trivially have $f(m,n,r) \leq \phi(m,n,K_r)$, so it suffices to show the reverse inequality. That is, we need to show that there exists a blowup construction meeting the desired bound.

Let $G_1, G_2, \ldots, G_m$ be graphs on $[n]$ with no double K_r and $\sum_{i=1}^m e(G_i) = \phi(m,n,K_r)$. Observe that any pair $\{i,j\} \subseteq [n]$ must be in some G_i, otherwise, we may add it to G_1 without creating a double K_r.

We call vertices v, v' *clones* if for all $u \in [n] \backslash \{v, v'\}$ and $i \in [m]$, the edge $\{u,v\} \in E(G_i)$ if and only if $\{u, v'\} \in E(G_i)$. Furthermore, we call $\{v, v'\}$ a *light edge* if $\{v, v'\}$ is in exactly one graph G_i.

We now apply Algorithm 1 to $G_1, G_2, \ldots, G_m$.

Algorithm 1 symmetrization algorithm

```
while ∃ a light edge whose endpoints are not clones do
    among all vertices incident to such an edge, select a vertex v with maximum degree
    B_v ← collection of vertices sending a light edge to v that are not clones of v
    while B_v ≠ ∅ do
        pick u ∈ B_v
        j ← colour of the light edge from u to v
        for 1 ≤ i ≤ m do
            if i ≠ j then;
                N_{G_i}(u) ← N_{G_i}(v)
            else if i = j then
                N_{G_i}(u) ← (N_{G_i}(v) \ {u}) ∪ {v}
            end if
        end for
    end while
end while
```

Claim 3. *Algorithm 1 terminates.*

Notice that at the end of the 'while $B_v \neq \emptyset$' loop, every vertex sending a light edge to v is a clone of v. This implies v along with the set L_v of vertices receiving light edges from v induce a clique of size at least two in some G_i, and an empty graph in every other graph G_j

with $j \neq i$. Moreover, any vertex $w \notin L_v$ sends edges to either all or none of the vertices in L_v, and if w is incident to L_v, then w sends edges to L_v in at least two graphs. It now follows that no light edge incident with a vertex in L_v will be picked again in an iteration of the out most while loop. Thus the algorithm can run through at most $n/2$ such iterations, and so it terminates.

Claim 4. *The resulting graphs $G'_1, G'_2, \ldots, G'_m$ do not contain a double K_r and $\sum_{i=1}^m e(G'_i) = \phi(m, n, K_r)$.*

Note that we replace u by a clone of v in the for loop of Algorithm 1. Since $\{u, v\}$ remains a light edge in this step, u and v cannot both belong to a double K_r in the modified graphs. Furthermore, any double K_r containing u after the for loop arises from a double K_r containing v prior to the for loop. But then $G_1, G_2, \ldots, G_m$ contained no double K_r to begin with, so $G'_1, G'_2, \ldots, G'_m$ do not contain a double K_r.

We now show that the algorithm does not reduce the number of edges. By our choice of v, we know $d(v) \geq d(u)$ for all $u \in B_v$ prior to the for loop. Hence, replacing u with a clone of v does not decrease the number of edge over a complete iteration of the inner while loop. Therefore, $\sum_{i=1}^m e(G'_i) = \phi(m, n, K_r)$. The proof of the claim is now complete.

Hence, the algorithm results in graphs $G'_1, G'_2, \ldots, G'_m$ with $\phi(m, n, K_r)$ edges and the additional property that light edges come in 'clone cliques.' We may thus partition the vertex set $[n]$ into k disjoint sets $V_1, V_2, \ldots, V_k$, such that each V_i induces a clique of light edges from the same graph. Moreover, for distinct $i, j \in [k]$, define S_{ij} to be the set of all edges between V_i and V_j, and note that any edge in S_{ij} appears in at least two modified graphs. The sets S_{ij} now yield a k-blowup. Notice that if the pattern of the k-blowup contains a double K_r, then the original graphs $G_1, G_2, \ldots, G_m$ must have contained a double K_r as well, contradiction. Thus the k-blowup is double K_r-free.

It remains to show that $k < R_M(K_r)$. For each edge $\{i, j\} \subseteq [k]$ in the pattern of the k-blowup, we assign an arbitrary distinct pair $\{a, b\} \subseteq L_{ij} \subseteq [m]$ to $\{i, j\}$. If $k \geq R_M(K_r)$, then there exists K_r in the pattern of the k-blowup colored by some distinct pair $\{a, b\} \subseteq [m]$. But then this implies the pattern of the k-blowup contains a double K_r, contradiction. This completes the proof. $\square$

3.4 Proof of Theorem 7

It is not hard to see that $\phi(2,n,K_3) = \binom{n}{2} + \lfloor n^2/4 \rfloor$: if G_1, G_2 is double triangle-free, then we have

$$e(G_1) + e(G_2) \le \binom{n}{2} + e(G_1 \cap G_2) \le \binom{n}{2} + \mathrm{ex}(n, K_3)$$

and so $\phi(2,n,K_3) \le \binom{n}{2} + \lfloor n^2/4 \rfloor$. Taking $G_1 = K_n$ and $G_2 = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ meets this bounds. The main result of this section is to show for all $n \ge 1$,

$$\phi(3,n,K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Let G_1, G_2, G_3 be double triangle-free. Define $H_k \subseteq G$ to be the graph with edges contained in at least k of the G_i's and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \le n^2/2$. Notice H_2 must not contain any triangles with two edges in H_3, so

$$e(H_2) + e(H_3) \le \binom{n}{2} + e(H_3) - |\{\{u,v\} : u \ne v, N_{H_3}(u) \cap N_{H_3}(v) \ne \emptyset\}|.$$

Let H_3' be the graph with the same vertex set as H_3 and edge set $\{\{u,v\} : u \ne v, N_{H_3}(u) \cap N_{H_3}(v) \ne \emptyset\}$. It suffices to show that $n/2 \ge e(H_3) - e(H_3')$.

Let $d_1 \ge d_2 \ge \cdots \ge d_n$ and $f_1 \ge f_2 \ge \cdots \ge f_n$ each be the degree sequence of H_3 and H_3', respectively. We show that $f_i \ge d_i - 1$ for all i. Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i. Let $S_i = |N_{H_3}(v_1) \cup \cdots \cup N_{H_3}(v_i)|$. Since

$$\sum_{u \in S_i} d_{H_3}(u) \ge d_1 + \cdots + d_i,$$

we have that $|S_i| \ge i$. But then $S_i \backslash \{u_1, \ldots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H_3'}(u) \ge d_i - 1$. Hence, $f_i \ge d_i - 1$ for all i, which yields

$$e(H_3') = \frac{1}{2}\sum_{i=1}^{n} f_i \ge \frac{1}{2}\sum_{i=1}^{n}(d_i - 1) = e(H_3) - \frac{n}{2}.$$

This proves Theorem 7. $\square$

4 Concluding Remarks

- For Theorem 1, we may not be able to achieve the same result with smaller n. For example, consider F to be the bowtie graph, i.e. the 5-vertex graph with two triangles sharing a vertex. The n-vertex extremal graph for F is given by $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ plus an edge when $n \geq 5$, otherwise it is the complete graph. For $n = 5$, the construction $G_1 = K_4, G_2 = K_5$ then shows that $\phi^*(2, 5, F) > 2 \cdot \mathrm{ex}(5, F)$. Fortunately, for non-bipartite F with $|V(F)| = k$, it is not hard to show $n \geq k^2$ is sufficient to avoid this issue.

- We note that Theorem 6 may be generalized to any family of non-bipartite graphs up to asymptotic error via Szemerédi's Regularity lemma

- One could ask for the analogous results for hypergraphs. That is, if F is an r-uniform hypergraph, let $\phi(m, n, F)$ be the maximum number of edges over m double F-free r-uniform hypergraphs on $[n]$. Again, we have $\phi(m, n, F) \geq \binom{n}{r} + (m-1) \cdot \mathrm{ex}(n, F)$. Another direction of generalization is to relax the constraint to no copies of F contained in the intersection of k of the graphs $G_1, G_2, \ldots, G_m$. Many of the theorems and proofs also hold in this case. For example, the proof of Theorem 4 applies for this generalization by merely changing the numbers.

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